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## Is an ss Complex a css Complex?\*

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In [1] Eilenberg and Zilber introduced *complete semisimplicial* (css) *complexes* (roughly speaking, graded sets with face and degeneracy operators). These turned out to be extremely useful in homotopy theory, in homological and homotopical algebra, and even in differential topology. They also considered (incomplete) *semisimplicial* (ss) *complexes* (no degeneracy operators), but these did not attract much attention as, until recently, all the ss complexes that turned up were either already complete or *could be completed* (i.e., without adding simplices, degeneracy operators could be added) *in a canonical manner*. Recently, however, Rourke and Sanderson ran into some really incomplete ss complexes. They pointed out [3] that the block bundle group  $\widetilde{PL}_q$  is an ss group which can indeed be completed, but *not* in a canonical manner.

It is clear that, in general, an ss complex cannot be completed at all (without adding simplices). That  $\widetilde{PL}_q$ , however, can be so completed is due to the fact that it satisfies the so-called *extension condition* [2, 3]. In fact we will prove.

**PROPOSITION.** *An ss complex  $X$  which satisfies the extension condition can be completed (although, in general, in many different ways).*

It is clear that any two such completions will have the same homotopy type.

**COROLLARY.** *An ss group can be completed (to a css complex) if and only if it satisfies the extension condition.*

This follows immediately from the fact that every css group satisfies the extension condition [2].

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*Proof of the Proposition.* Let  $F$  denote the forgetful functor which assigns to a css complex the underlying ss complex. Then we have to construct a css complex  $W$  and an *isomorphism*  $g : FW \rightarrow X$ . This is done as follows. Let  $W^{-1}$  denote the empty css complex. Then the unique map  $g^{-1} : FW^{-1} \rightarrow X$  is clearly 1-1 and is onto in dimensions  $< 0$ . Now assume we already constructed a css complex  $W^{n-1}$  and a map  $g^{n-1} : FW^{n-1} \rightarrow X$  which is 1-1 and is onto in dimensions  $< n$ . Let  $W^n$  denote the css complex obtained from  $W^{n-1}$  by attaching a non-degenerate  $n$ -simplex for every  $n$ -simplex of  $X$  which is not in the image of  $g^{n-1}$ , and let  $Y^n \subset FW^n$  be the subcomplex consisting of  $FW^{n-1}$  and the new  $n$ -simplices. Then  $g^{n-1}$  extends to a map  $h^n : Y^n \rightarrow X$  which is 1-1 and is onto in dimensions  $\leq n$ . Moreover, the geometric realization  $|Y^n|$  is a (strong deformation) retract of  $|FW^n|$  and [3, 1.3], as  $X$  satisfies the extension condition,  $h^n$  can thus be extended to a map  $g^n : FW^n \rightarrow X$  which one readily verifies to be again 1-1 and to be onto in dimensions  $\leq n$ . The desired isomorphism  $g : FW \rightarrow X$  one now gets by putting  $W = \bigcup W^n$  and  $g = \bigcup g^n : FW \rightarrow X$ .

## REFERENCES

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